Spectral Analysis

A signal x may be represented as a function of time as $x(t)$ or as a function of frequency $X(f)$. This is due to relationships developed by a French mathematician, physicist, and Egyptologist, Joseph Fourier(1768-1830). Both the Fourier transform and the closely associated Fourier series are named in his honor. Even the telegraph hadn't been invented in his lifetime and were he alive today he would be astonished at the number of algorithms, software, and electronic test instruments that bear his name. The fact that he lived to accomplish the foundation of spectral analysis is miraculous since he was the last of eighteen children and escaped the guillotine several times during the French Revolution.

Joseph Fourier

The two representations of a signal are connected via the Fourier transform

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)dt$$

Many of the signals of interest in electrical engineering are periodic functions of time. A periodic function is one for which

$$x(t) = x(t \pm nT)$$

where n is any integer and T is the smallest interval of time for which this relationship is true. The interval T is called the period of the periodic function. If the periodic function satisfies constraints known as the Dirichlet conditions (which are satisfied by any function produced by nature) it may be expanded in a Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \exp(jn\omega_p t)$$

where $\omega_p = 2\pi f_p$ and $f_p = 1/T$ is the frequency in Hertz of the periodic function. This is known as the complex Fourier series representation of a periodic function. The expansion coefficient c_n are complex constants which can be determined from $x(t)$ as

$$c_n = \frac{1}{T}\int_{\alpha}^{\alpha+T} x(t)\exp(-jn\omega_p t)dt$$

where α is any real number. The terms in the Fourier series for which n is an even integer are known as the even harmonics and the terms for which n is an odd integer are known as odd harmonics. The term for which $n = \pm 1$ are known as the fundamental. Alternative representation of the Fourier series is the real trigonometric series

$$x(t) = a_o/2 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_p t) + b_n \sin(n\omega_p t)]$$

where

$$a_n = \frac{2}{T}\int_{\alpha}^{\alpha+T} x(t)\cos(n\omega_p t)dt \qquad b_n = \frac{2}{T}\int_{\alpha}^{\alpha+T} x(t)\sin(n\omega_p t)dt$$

These are not different series; just two ways of expressing the same result. The expansion coefficients are related by

$$c_n = \frac{a_n - jb_n}{2} \qquad a_n = 2\,\mathrm{Re}(c_n) \qquad b_n = -2\,\mathrm{Im}(c_n)$$

The Fourier transform of a periodic function is then given by

$$X(f) \quad = \quad \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi f t)dt =$$

$$\int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_n \exp(jn\omega_p t)\exp(-j2\pi f t)dt = \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_p)$$

which is a line spectra. The function δ is the Dirac delta which makes the spectra zero everywhere excepts at frequencies which are integral multiples of f_p. The lines have amplitudes or weights of c_n. If a plot is made of the magnitude of the spectra for only positive frequencies it would consists of lines at $f = nf_p$

and the height of each line would be $2|c_n|$ If the spectra is to be plotted in rms each line would be $\sqrt{2}|c_n|$.

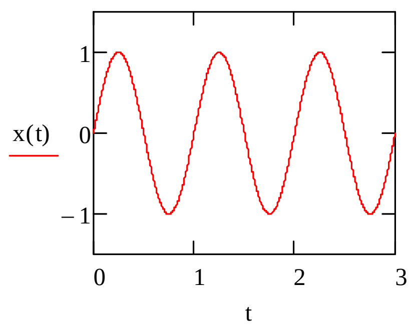
A topic tangential to Fourier or Spectral analysis is Total Harmonic Distortion (THD) which measures how much a signal differs from a perfect sine wave. It is defined as (in percent) as

$$THD = 100\sqrt{\sum_{n=2}^{\infty}\left|\frac{c_n}{c_1}\right|^2}$$

Sine Wave

A sine wave with amplitude A and frequency $f_p = 1/T$ is given by

$$x(t) = A\sin(\omega_p t)$$



Sine Wave

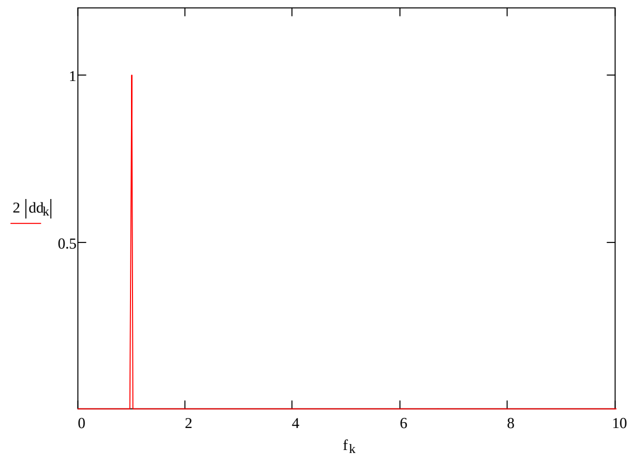
is particularly simple since

$$\sin\theta = \frac{\exp(j\theta) - \exp(-j\theta)}{2j}$$

so

$$c_n = \begin{cases} \frac{1}{2j} & n = 1 \\ -\frac{1}{2j} & n = -1 \\ 0 & n \neq \pm 1 \end{cases}$$

and the spectra is given by



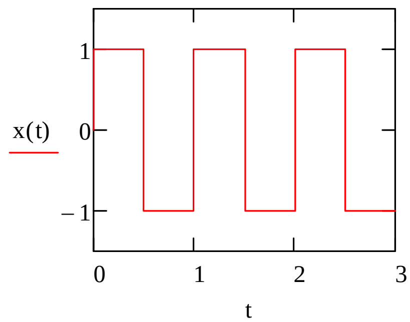
Spectra of Sine Wave.

where the frequencies and amplitudes have been normalized to unity for simplicity. So the Fourier series representation of a perfect sine wave is a perfect sine wave. Which makes the $THD = 0$ which means that there is no harmonic distortion or, another way of putting it, nothing looks like a sine wave more than a sine wave.

Square Wave

A symmetric square wave with a dc level of zero is one which is $+A$ half the time and $-A$ the other half. The choice of the time origin is arbitrary by a common one is

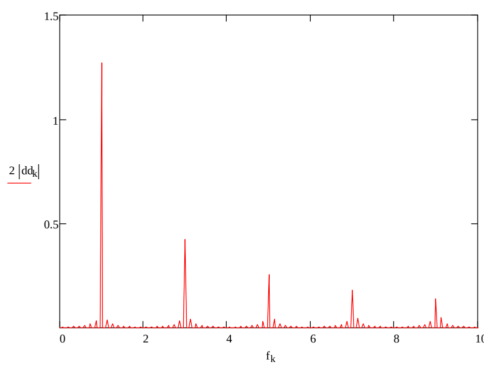
$$x(t) = \begin{cases} -A & -T/2 < t < 0 \\ +A & 0 < t < T/2 \\ x(t \pm nT) & \text{elsewhere} \end{cases}$$



Symmetric Square Wave.

where $A = 1$ and $T = 1$ in the figure. The complex Fourier expansion coefficients are

$$c_n = \begin{array}{ll} \frac{4}{\pi}\frac{1}{n} & n \text{ odd} \\ 0 & n \text{ even} \end{array}$$



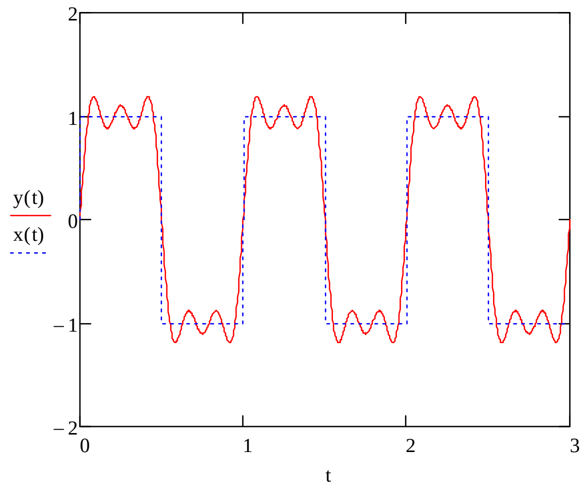
Spectra of Symmetric Square Wave

The normalized spectra is

$$\frac{c_n}{c_1} = \begin{array}{ll} \frac{1}{n} & n \text{ odd} \\ 0 & n \text{ even} \end{array}$$

As a comparison of how well the Fourier series represents a square wave a plot can be made of the square wave and the first five harmonics

$$x(t) = a_o/2 + \sum_{n=1}^{5} [a_n \cos(n\omega_p t) + b_n \sin(n\omega_p t)]$$



Square Wave and Fourier Approximation using First 5 Terms.

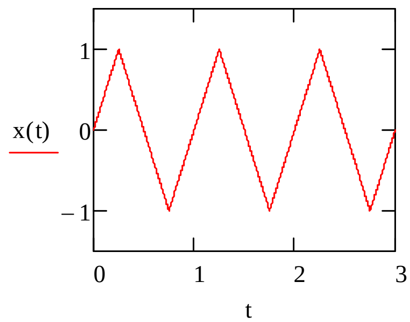
The ringing that occurs where the square wave is switching levels is known as the Gibbs phenomenon. Using the first 9 components the THD for the square wave is 42.879% which simply means a square wave doesn't look very much like a sine wave.

Triangular Wave

A symmetric triangular wave consists of alternating straight lines with slopes of equal magnitudes and a dc level of zero.

$$x(t) = \begin{cases} 4A\frac{t}{T} & 0 \le t \le T/4 \\ -4A\frac{t}{T} + 2A & T/4 \le t \le 3T/4 \\ 4A\frac{t}{T} - 4A & 3T/4 \le t \le T \\ x(t \pm nT) & \end{cases}$$

where A is the amplitude and T the period of the triangular wave.



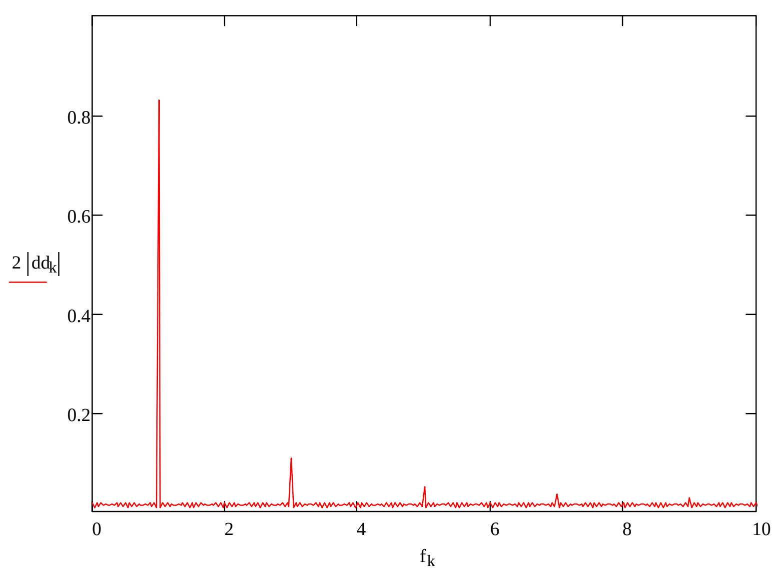
Symmetric Triangular Wave.

The complex Fourier expansion coefficients are

$$c_n = -jA\frac{4\sin\frac{n\pi}{2}}{\pi^2 n^2}$$

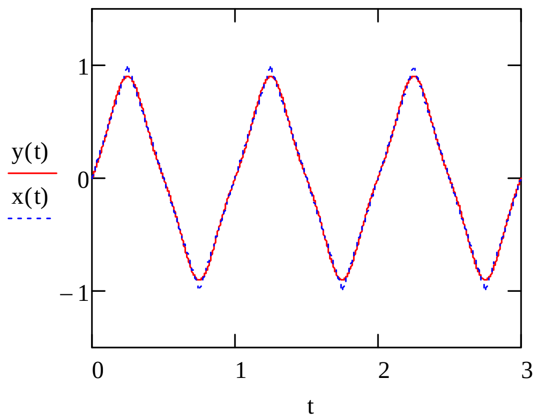
which are zero for n even and roll off as $1/n^2$ for n odd.

The spectra for the triangular wave is



Spectra for Symmetric Triangular Wave.

A plot of the triangular wave and the first 3 components shows they are almost indistinguishable



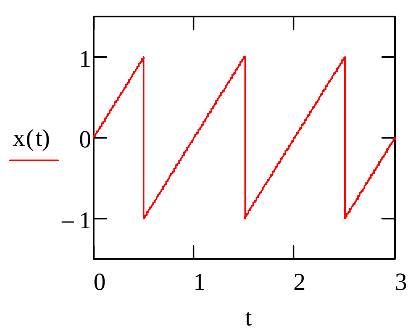
Triangular Wave and Approximation by First 3 Components.

The THD is only 12.048% which means that a triangular wave is reasonable close to a sine wave.

Ramp

A ramp or sawtooth wave is one for which

$$x(t) = \begin{cases} 2A\frac{t}{T} & -\frac{T}{2} < t \leq \frac{T}{2} \\ x(t \pm nT) & n \text{ any integer} \end{cases}$$

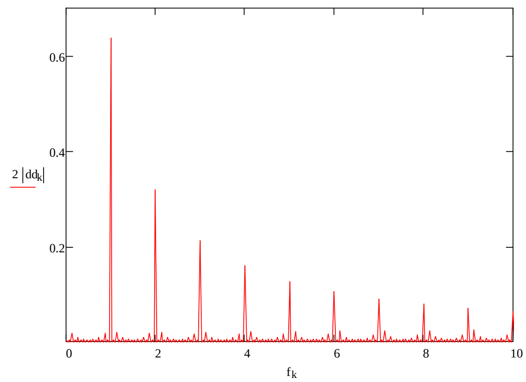


Ramp Wave.

The expansion coefficients are

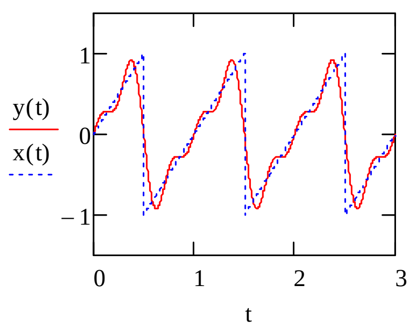
$$c_n = jA\frac{\cos(n\pi)}{n\pi}$$

The spectra is given by



Spectra of Ramp.

Using the first 3 components the approximation and the ramp are



Ramp and Approximation.

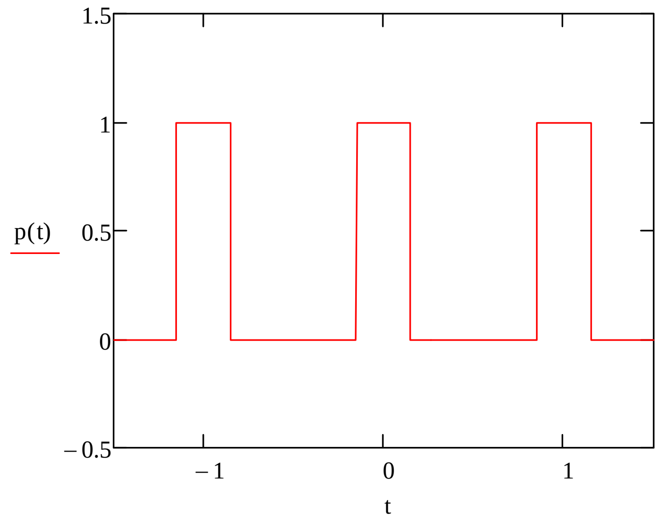
The THD using the first 9 components is 75.469%.

Rectangular Pulse Train

A rectangular pulse train is similar to a square wave in that it switches between two levels but the duty cycle is not 50%. The duty cycle is the percentage of the time the waveform is in the high state. The pulse train is

$$x(t) = \begin{cases} A & |t| \leq \frac{\tau}{2} \\ 0 & \frac{\tau}{2} < t < \frac{T}{} \\ x(t \pm nT) & n \text{ any integer} \end{cases}$$

so the duty cycle is $d = \tau/T$.

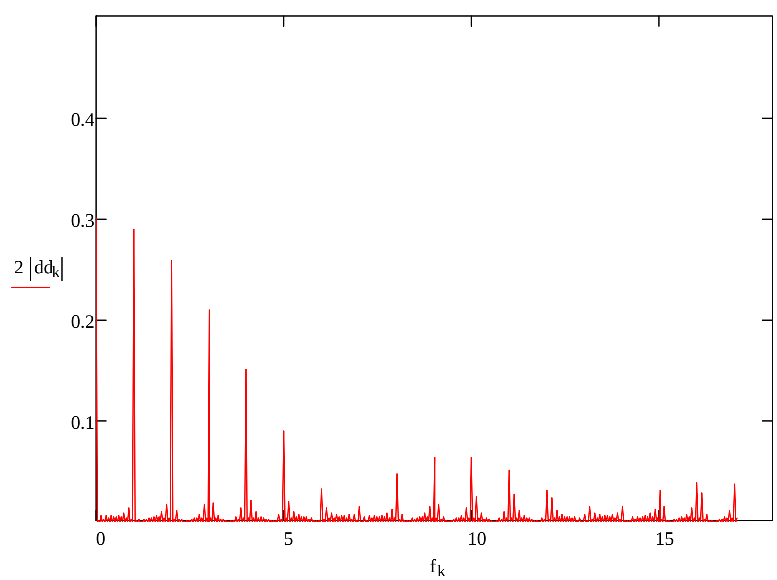


Pulse Train with Duty Cycle 0.15.

The Fourier expansion coefficients are

$$c_n = Ad\frac{\sin(\pi nd)}{\pi nd}$$

with spectra

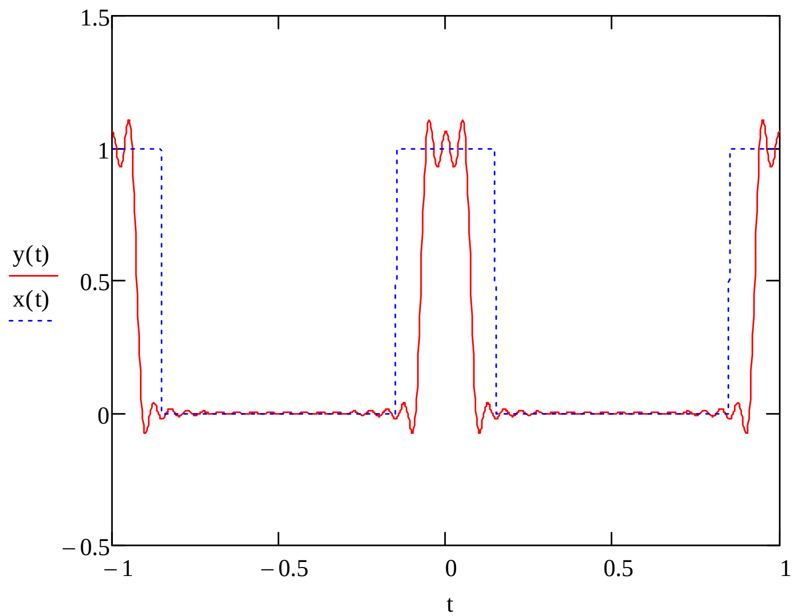


Spectra of Pulse Train

which is, of course, a line spectra but the envelope of the spectra has a $\sin(x)/x$

behavior.

The approximation of the pulse train as the first 20 terms of the Fourier series is



Approximation of pulse train as first 20 Terms of Fourier Series.

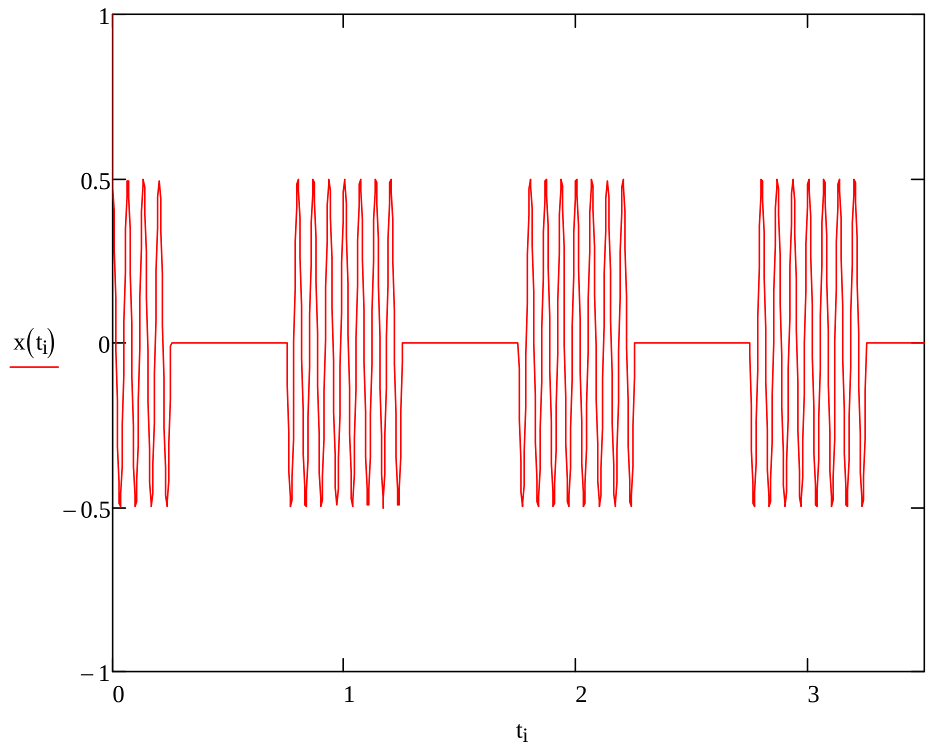
for which the THD is 139 which means this really doesn't look like a sine wave. If the duty cycle $d = 0.5$ this becomes a symmetric square wave.

RF Pulse Train

A rf pulse train is a rectangular pulse train multiplied to a sinusoidal with a frequency much higher than that of the train. Mathematically it is given by

$$x(t) = \begin{cases} A\cos(\omega_c t) & |t| \le \frac{\tau}{2} \\ x(t \pm nT) & n \text{ is an integer} \end{cases}$$

The duration of the pulse is τ. It is assumed that f_c is an integral multiple of $1/T$. The number of cycles in the pulse $N = \tau f_c$ which is assumed to be an integer. This is the sort of signal used in radar.



RF Pulse Train.

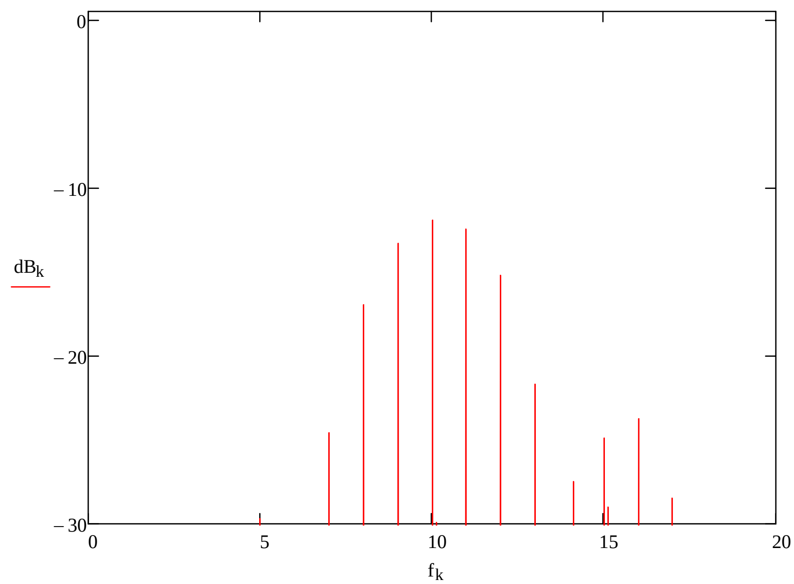
The Fourier expansion coefficients are given by

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\exp(j2\pi f_p nt)dt = \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\cos(2\pi f_c t)\exp(j2\pi f_p nt)dt =$$

$$\frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\cos(2\pi f_c t)\cos(2\pi f_p nt)dt =$$

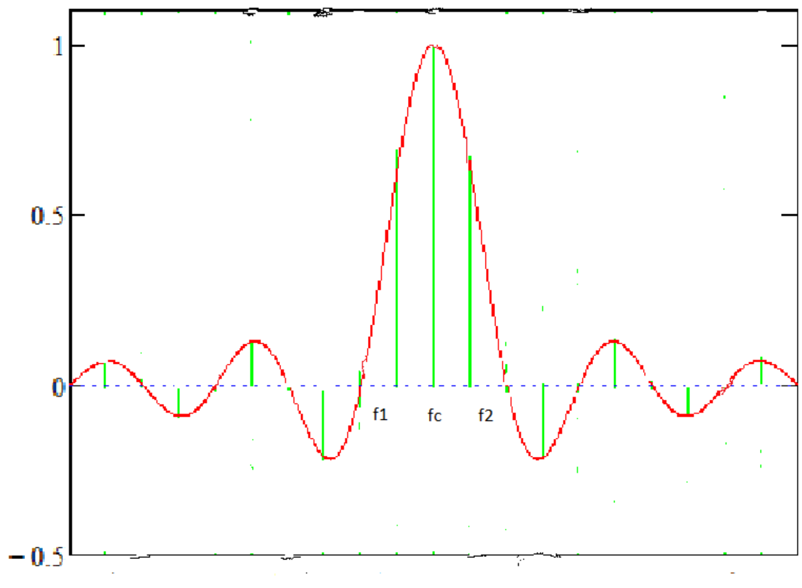
$$\frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\cos(2\pi f_c t)\cos(2\pi f_p nt)dt = \frac{A\tau}{2T}\left[\frac{\sin\pi(f_c - nf_p)}{\pi(f_c - nf_p)} + \frac{\sin\pi(f_c + nf_p)}{\pi(f_c + nf_p)}\right]$$

which shows that the spectra of the rf pulse train is just that of the rectangular pulse train shifted up to $f = f_c$ and down to $f = -f_c$, The spectra is



RF Pulse Train Spectra.

The distance f_c to the first null is $\tau/2$ where $\tau = N/f_c$ where N is the number of cycles in the rf pulse. So the spectra of all of these signals is a line spectra which is a direct consequence of their periodicity. However, the value of the expansion coefficients is a function of the shape. The envelope of the spectra has the typical $\sin(u)/u$ shape.



Spectra of RF Pulse Centered about High Frequency Carrier.

The spectra has nulls centered about the carrier at frequencies

$$f_1 = f_c - 1/\tau \underline{\qquad\qquad\qquad} f_2 = f_c + 1/\tau$$

so the difference between the first two nulls about the center of the $\sin(u)/u$ is given by

$$\Delta f = 2/\tau = 2f_c/N$$

where N is the number of cycles in the rf pulse.

The spectra of a signal is important for a number of reasons. Most importantly it determines the bandwidth that would have to be used to pass or transmit the signal without distortion. It is fundamental in signal processing and telecommunications.