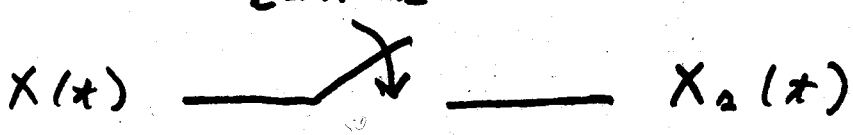
Spectral Analysis – Frequency Content

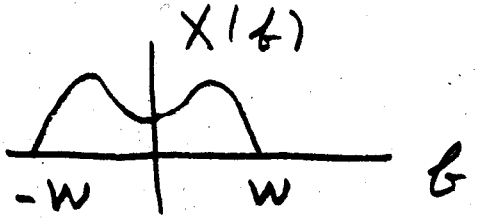
$$x(t) \longleftrightarrow X(f)$$

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$



$T_s \equiv$ sampling period

$f_s = \frac{1}{T_s} =$ sampling rate



$f_s > 2W$ Nyquist criterion to avoid aliasing

If N points are used for the FFT

$$\Delta f = \frac{f_s}{N}$$

Want Δf, the frequency resolution to be small.

Fourier Series

If $x(t)$ is periodic with period T, then $x(t)$ can be expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} \bar{c}_n e^{jn\omega_p t}$$

where $\omega_p = 2\pi f_p$ and $f_p = 1/T$.

This is known as the (complex) Fourier series representation of $x(t)$. The expansion coefficients can be obtained from

$$\bar{c}_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_p t} dt$$

This representation is identical to the real trignometric Fourier series

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\omega_p t + b_n \sin n\omega_p t]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)^{\frac{1}{2}} \sin [n\omega_p t + \phi_n] =$$

$$\bar{c}_o + \sum_{n=1}^{\infty} 2\,|\bar{c}_n| \cos [n\omega_p t + \angle \bar{c}_n]$$

where $\phi_n = \tan^{-1} \frac{a_n}{b_n}$ and $\angle \bar{c}_n = \phi_n - \pi/2$.

$$a_n = \frac{2}{T} \int_0^T x(t) \cos n\omega_p t\, dt \quad b_n = \frac{2}{T} \int_0^T x(t) \sin n\omega_p t\, dt$$

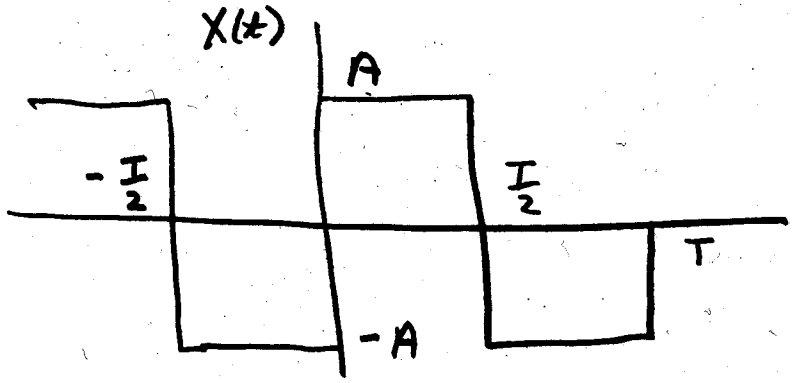
The expansion coefficients are related by

$$\bar{c}_n = \frac{a_n - jb_n}{2} \quad a_n = 2Re\{\bar{c}_n\} \quad b_n = -2Im\{\bar{c}_n\}$$

The frequency content or spectra of $x(t)$ is given by the Fourier Transform, $X(f)$, of $x(t)$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \sum_{-\infty}^{\infty} \bar{c}_n \delta(f - nf_p)$$

which is a line spectra.



$$f_p = \frac{1}{T}$$

$$\omega_p = 2\pi f_p$$

$$X(t) = \sum_{n=-\infty}^{\infty} \bar{c}_n e^{jn\omega_p t} \quad ; \quad \bar{c}_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t) e^{-jn\omega_p t} dt$$

$$\bar{c}_n = \frac{1}{T}\left[\int_{-\frac{T}{2}}^{0} (-A) e^{-jn\omega_p t} dt + \int_{0}^{\frac{T}{2}} A e^{-jn\omega_p t} dt\right]$$

1st integral let $u = -t \Rightarrow dt = -du$, exchange upper & lower limits of integration, combine and use

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\bar{c}_n = \frac{-2jA}{T}\int_0^{\frac{T}{2}} \sin n\omega_p t\, dt = \frac{2jA}{n\omega_p T} \cos n\omega_p t \Big|_0^{\frac{T}{2}}$$

$$\bar{c}_n = \frac{2jA}{2\pi n}(\cos n\pi - 1) = \begin{cases} 0 & n \text{ even} \\ -\frac{4jA}{2\pi n} & n \text{ odd} \end{cases}$$

$$a_n = 2\,Re\{\bar{c}_n\} = 0 \quad , \quad b_n = -2\,Im\{\bar{c}_n\} = \frac{4A}{n\pi}$$

$$X(t) = \frac{4A}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin n\omega_p t}{n}$$

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****     CIRCUIT DESCRIPTION

******************************************************************************

vi 1 0 pwl(0 -1 1f 1 0.5 1 0.50001 -1 1 -1)
*vi 1 0 pulse(-1 1 0 1f 1f 0.5 1)
r 1 0 1
.tran 1m 1
.four 1 v(1)
.probe

 fourier analysis of square wave

****     FOURIER ANALYSIS                    TEMPERATURE =   27.000 DEG C

******************************************************************************

FOURIER COMPONENTS OF TRANSIENT RESPONSE V(1)

 DC COMPONENT =  -1.001001E-03

HARMONIC     FREQUENCY     FOURIER     NORMALIZED     PHASE         NORMALIZED
   NO          (HZ)       COMPONENT    COMPONENT      (DEG)      PHASE (DEG)

    1      1.000E+00     1.273E+00     1.000E+00   -9.009E-02     0.000E+00
    2      2.000E+00     2.002E-03     1.572E-03   -9.018E+01    -9.009E+01
    3      3.000E+00     4.244E-01     3.333E-01   -2.703E-01    -1.802E-01
    4      4.000E+00     2.002E-03     1.572E-03   -9.036E+01    -9.027E+01
    5      5.000E+00     2.547E-01     2.000E-01   -4.505E-01    -3.604E-01
    6      6.000E+00     2.002E-03     1.572E-03   -9.054E+01    -9.045E+01
    7      7.000E+00     1.819E-01     1.429E-01   -6.306E-01    -5.405E-01
    8      8.000E+00     2.002E-03     1.572E-03   -9.072E+01    -9.063E+01
    9      9.000E+00     1.415E-01     1.111E-01   -8.108E-01    -7.207E-01
 n        n fp      sqrt(an^2+bn^2)   |cn/c1|   phi_n=tan^-1 an/bn   phi_n - phi_1
  TOTAL HARMONIC DISTORTION =  4.288100E+01 PERCENT
```



$$x(t) = X_{DC} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}\, \sin\left[n\omega_p t + \phi_n\right]$$

$$THD = \text{total harmonic distortion} = 100\, \frac{\sqrt{\sum_{n=2}^{\infty} |\bar{c}_n|^2}}{|\bar{c}_1|}$$

For the square wave, using the first 9 terms

$$THD = 100 \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{1}{9}\right)^2} = 42.88\,\%$$

Square Wave $A := 10$ $T := 1$ $N := 9$ $j := \sqrt{-1}$ $n := 1 .. N$ $f_p := \frac{1}{T}$

$$x(t) := \begin{cases} (-A) \text{ if } \frac{-T}{2} \le t \le 0 \\ A \text{ if } 0 < t < \frac{T}{2} \\ 0 \text{ otherwise} \end{cases}$$

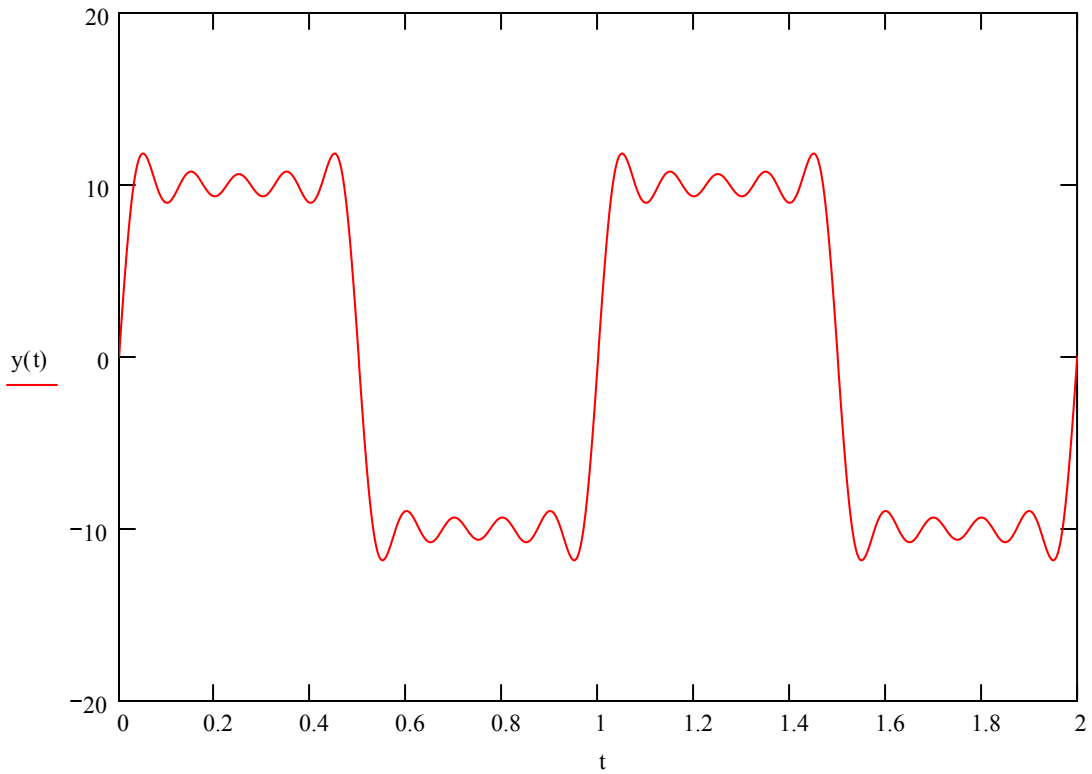
$$c_n := \frac{1}{T} \cdot \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t) \cdot e^{-j \cdot 2 \cdot \pi \cdot f_p \cdot n \cdot t} dt \qquad d_n := \frac{c_n}{c_1}$$

$c_n =$	$d_n =$	$2 \cdot \lvert c_n \rvert =$
-6.366i	1	12.732
0	0	0
-2.122i	0.333	4.244
0	0	0
-1.273i	0.2	2.546
0	0	0
-0.909i	0.143	1.819
0	0	0
-0.707i	0.111	1.415

$a_n := 2 \cdot \mathrm{Re}(c_n)$ $b_n := -2 \cdot \mathrm{Im}(c_n)$

$$y(t) := \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cdot \cos(n \cdot 2 \cdot \pi \cdot t \cdot f_p) + b_n \cdot \sin(n \cdot 2 \cdot \pi \cdot t \cdot f_p) \right)$$

$$THD := 100 \cdot \sqrt{\sum_{n=2}^{N} \left(\lvert d_n \rvert \right)^2} \qquad THD = 42.879$$



Spectral Analysis of Square Wave Using the FFT

Define One Cycle p(t) of Square Wave

$$p(t) := \begin{cases} (-A) & \text{if } \frac{-T}{2} < t < 0 \\ A & \text{if } 0 \le t \le \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

Define Square Wave x(t) for 5 Cycles

$$xx(t) := \sum_{i=0}^{10} p(t - i \cdot T)$$

The initial time is tstart $t_{start} := 0$ The final time is tstop $t_{stop} := 10$

The FFT requires that the number of points for the DFT be a power of 2, so pick

$K := 10$ $N := 2^K$ $N = 1024$ which makes the sampling period $\Delta t := \frac{t_{stop} - t_{start}}{N}$

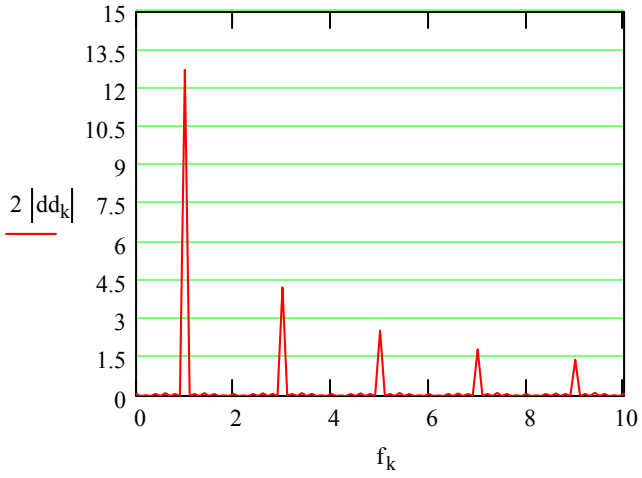
and the sampling rate $f_s := \frac{1}{\Delta t}$ $f_s = 102.4$ $i := 0 .. N - 1$

The sampled values are $s_i := xx(t_{start} + i \cdot \Delta t)$ which is an array with N points

The FFT is then $dd := FFT(s)$ $M := last(dd)$ $k := 0 .. M$

which is a complex array with M+1 points. $M = 512$

The frequency resolution is $\Delta f := \frac{f_s}{N}$ and the frequency $f_k := k \cdot \Delta f$ which yields



Which is Reasonable Accurate for the lower frequency components but drops off at high frequencies due to aliasing

$\Delta f = 0.1$