ECE 3043 Short Notes

A Linear System's Frequency Response and Plotting it in Logarithmic Scale

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You work with a circuit's *transfer function* all the time in this course and make extensive use of the *logarithmic scale*: you are asked to Bode-plot a system's *frequency response* in your homework assignments, lab reports, or maybe exams using Matlab, Agilent VEE, or even with bare hands, where the horizontal axis reads *Frequency (log-scale)*. Good or bad, you will continue to see such plots here and there if you end up working as an electrical engineer or applied physicist, especially in the field of electronics or systems control. In fact, logarithmic scale will prove helpful to you wherever you are dealing with large numbers or working with data over a large range. You should hence know well the steps involved in finding a linear system's transfer function and converting it to its frequency response, and should also feel comfortable plotting data vs. log-scale or reading data off of a chart with either axis (or both!) in log-scale. The purpose of this short note is to create a better feeling about the concepts of transfer function, frequency response, and logarithmic scale through an example. The example is a very simple one because the focus is intended to be on the *approach* for analyzing such a problem in a systematic way.

Let's consider a simple RC filter: A resistor of resistance R in series with a capacitor with capacitance C, where the output is the voltage across the capacitor (Figure 1). The **transfer function** for this *"linear"* circuit in the Laplace domain is

$$T(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + RCs}. \tag{1}$$

(The reason I emphasize on the linearity of the circuit is that you can have transfer function only for a linear system or, more carefully put, for a system only in its linear regime of operation.) The variable s is a complex frequency, i.e. it has both real and imaginary parts: $s = \sigma + i\omega$, where $i = \sqrt{-1}$. (Mathematicians and physicists like to denote the square root of -1 by i, but since the letter i is already taken up for current in electrical engineering, electrical engineers have adopted j. Since I hold physicists in high esteem, let me use i.)

To go from the Laplace domain to the Fourier domain, we need not care about the real part of s and should merely replace it by $i\omega$ yielding

$$T(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{1}{1 + i\omega RC}. \tag{2}$$

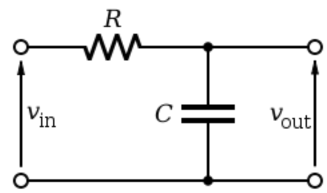

Figure 1: A simple low-pass filter

The transfer function for our simple circuit is now ready for manipulations as a function of the radian frequency ω in the Fourier domain.

What we have at hand is a fraction in terms of complex values as the denominator has both real and imaginary parts. So, before going any further, let's review very quickly what we need to remember about complex numbers at this point. Every complex number z could be written in, at least, two equivalent forms: in terms of its real and imaginary parts, namely $z = x + iy$, or in terms of its amplitude and phase, i.e. $z = re^{i\theta}$. For obvious reasons I will call the former the *real-imaginary component representation* and the latter the *magnitude-phase representation* of the complex number z. Clearly, the pair $x-y$ and the pair $r - \theta$ of the two representations are related by

$$x = r\cos\theta \qquad y = r\sin\theta \tag{3a}$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan\frac{y}{x}, \tag{3b}$$

where I have exploited the so-called Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$.

Our goal here is to find the magnitude-phase representation of the transfer function of Eqn.(2). Although we can easily go directly to this desired form, let's do it step by step, going first from the transfer function to the real-imaginary component representation and from there to the magnitude-phase one. For that, multiply both the numerator and the denominator of Eqn.(2) by the *complex conjugate* of its *denominator*:

$$\begin{aligned} T(\omega) &= \frac{1}{1+i\omega RC} = \frac{1}{1+i\omega RC}.\frac{1-i\omega RC}{1-i\omega RC} = \frac{1-i\omega RC}{1+(\omega RC)^2} \\ &= \frac{1}{1+(\omega RC)^2} + i\frac{-\omega RC}{1+(\omega RC)^2}. \end{aligned} \tag{4a}$$

The real and imaginary parts of the transfer function are now known. It is straightforward then to find the magnitude-phase representation using Eqn.(3b) as

$$T(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{1}{\sqrt{1+(\omega RC)^2}} e^{i[\arctan(-\omega RC)]}. \tag{4b}$$

So the magnitude and phase of the transfer function are

$$\text{Magnitude}: \frac{1}{\sqrt{1+(\omega RC)^2}}, \tag{6a}$$

$$\text{Phase}: \arctan(-\omega RC) = -\arctan(\omega RC). \tag{6b}$$

In the last line, I have used the fact that $\arctan(x)$ is an odd function of x, for real x.

Question: How could we find these directly, i.e. without having to find the real-imaginary component representation?

Answer: Think of Magnitude and Phase as *operators*. Let's represent the magnitude operator by $|\cdot|$, such that the magnitude of the complex number z is $|z| = r$, and represent the phase operator by $\angle\cdot$, such that the phase of z is $\angle z = \theta$. How the magnitude operator works is quite simple: the magnitude of the product of two complex numbers is the product of their magnitudes, i.e. for two complex numbers z_1 and z_2, $|z_1 z_2| = |z_1||z_2|$. On the other hand, how the phase operator treats the product of two complex numbers z_1 and z_2 is similar to a logarithm, i.e. just as $\log(z_1 z_2) = \log(z_1) + \log(z_2)$, we have $\angle z_1 z_2 = \angle z_1 + \angle z_2$. As you may expect, $\angle(1/z) = \angle z^{-1} = -\angle z$ so that $\angle(z_1/z_2) = \angle z_1 - \angle z_2$. Now let's apply these operators to our transfer function.

$$\begin{aligned}|T(\omega)| &= \left|\frac{V_{out}(\omega)}{V_{in}(\omega)}\right| = \left|\frac{1}{1+i\omega RC}\right| = \frac{1}{|1+i\omega RC|} \\ &= \frac{1}{\sqrt{1+(\omega RC)^2}} \end{aligned} \tag{7a}$$

$$\begin{aligned}\angle T(\omega) &= \angle\frac{V_{out}(\omega)}{V_{in}(\omega)} = \angle\frac{1}{1+i\omega RC} = \angle 1 - \angle(1+i\omega RC) = 0 - \arctan(\omega RC) \\ &= -\arctan(\omega RC)\end{aligned} \tag{7b}$$

These are the exact same results of Eqn.(6), obtained previously through finding the magnitude-phase representation of our system's transfer function (after finding its real-imaginary component representation!). The combination of the magnitude and the phase response of a system is called its **frequency response**. So, we have thus far found exact expressions for the frequency response of our simple RC circuit with the input and output as specified in Figure 1. Please note that both the magnitude and the phase response are dimensionless quantities. The angular frequency ω has units of radians per second and the product RC has dimensions of time and if R is in Ohms and C is in Farads, would be in seconds. The product ωRC is hence $rad/sec \times sec = rad$, i.e. merely a number. (Remember that radian is itself a dimensionless quantity because it is the ratio of two lengths, namely the length of an arc and its radius.)

These expressions could be plotted vs. the frequency in this current form, without any further manipulation. That would be the frequency response in linear scale vs. frequency, again, in linear scale. I suggest that you try doing that in MATLAB for $R = 1k\Omega$ and $C = 1nF$. The problem, you will readily realize, is that your horizontal axis should be very long in order for your plot to show a complete view of the magnitude and phase responses. For instance, if your horizontal axis ω covers values from 10^4 upto 10^5, i.e. an interval of $9 \times 10^4 rad/sec$, you will barely notice any interesting and important feature on the curve. (In fact, as you will see shortly, a plot that is just *good enough* should cover from $10^4 rad/sec$ to $10^8 rad/sec$!) So, we need a way to somehow fit in our large horizontal axis. Search through different mathematical functions you know. We want a function that gets a large number as input and returns a smaller number as output. We want it to give us a larger output for a larger input, because we do not want it to change the order of the numbers on our horizontal axis. (It would be bad if your frequency axis gets messed up!) If you add to these the requirement of simplifying calculations too, then you will vividly see that logarithm is an ideal choice. Its output is just minuscule compared to the corresponding input, e.g. $\log_{10}(10^4) = 4$ which is only % 0.04 of the input. It is also monotonic, i.e. the relationship $10^5 > 10^4$ for two inputs is preserved between their logs ($\log_{10}(10^5) = 5 > \log_{10}(10^4) = 4$). Finally, it converts multiplication to summation, division to subtraction, and an exponent to a coefficient, leading to significant simplification of calculations. As for the base of the logarithm, we will naturally pick 10.

Now let's rewrite the magnitude response in a weird way. (You will realize why I am doing this in a minute.) I can write ω as $10^{\log_{10}(\omega)}$; remember that taking the log and exponential are inverse of each other such that $\omega = 10^{\log_{10}(\omega)}$ is a trivial identity. Let's pick a name for $\log_{10}\omega$, say x: $x = \log_{10}\omega$. So $\omega = 10^{\log_{10}(\omega)} = 10^x$. The transfer function of Eqn.(7a), as a function of x, then, is

$$|T(x)| = \left|\frac{V_{out}(x)}{V_{in}(x)}\right| = \frac{1}{\sqrt{1 + (RC \times 10^x)^2}}. \tag{8}$$

Looks like I made it less friendly. To make things even worse, let's square both sides, then take their base-10 log and finally multiply them by 10.

$$|T(x)|_{\text{dB}} = 10 \times \log_{10}\left|\frac{V_{out}(x)}{V_{in}(x)}\right|^2 = 10 \times \log_{10} \frac{1}{1 + (RC \times 10^x)^2}. \tag{9}$$

(This is the familiar procedure for finding the amplitude response in dB, hence the notation $|T(x)|_{dB}$. You do $10 \times \log_{10} |\cdot|^2$ or, equivalently, $20 \times \log_{10} |\cdot|$, where the vertical lines represent the absolute value and where you replace the $\cdot$ with the transfer function expression.) I dragged you along manipulating Eqn.(7a) to find this: **The expression of Eqn.(9) is what MATLAB plots as the magnitude response when you use the command `bodeplot`**. If you rewrite the phase response, i.e. Eqn.(7b), as a function of x, then you will have the expression for the phase response that `bodeplot` plots in MATLAB. Note that you do not take the final step of applying $10 \times \log_{10} |\cdot|^2$ to your expression when dealing with the phase response. Both the magnitude and the phase plots are in logarithmic scale meaning that their horizontal axis is $\log_{10}\omega$ (or our x!). The vertical axis for the magnitude response is in dB, which is itself log-scale, while the vertical axis for the phase response is linear, either in degrees or radians. The Bode plot of the frequency response of our simple circuit is shown in Figure 2. As evident from this plot, it is a low-pass filter.

To make more sense of the expression of Eqn.(9), let's examine it at two extremes of low and high frequencies. But *how low*, or *how high*? Going back to Eqn.(7a), if the frequency ω is so low that we can *neglect* the term $(\omega RC)^2$ as compared to the term 1 that is being *added* to it in the denominator, then we can simplify our magnitude response:

$$\text{Low Frequencies}: \lim_{\omega RC \ll 1} |T(\omega)| = \lim_{\omega RC \ll 1} \frac{1}{\sqrt{1+(\omega RC)^2}} = \frac{1}{1} = 1. \tag{10}$$

Similarly, if the frequency is so high that $(\omega RC)^2$ is much larger than the term 1 that is being added to it in the denominator, then we can neglect the 1 and find a simplified expression:

$$\text{High Frequencies}: \lim_{\omega RC \gg 1} |T(\omega)| = \lim_{\omega RC \gg 1} \frac{1}{\sqrt{1+(\omega RC)^2}} = \frac{1}{|\omega RC|}. \tag{11}$$

(We can usually safely do this act of *neglecting* one number *added* to another, when the first is at least one order of magnitude smaller than the second. This is not a general statement, of course, but holds true for most cases in this course.) Since $\omega = 10^x$, the same recipe could readily be followed for the expression used by MATLAB to plot Bode diagrams, i.e. Eqn.(9), leading to

$$|T(x)|_{\text{dB, Low Frequency}} = 10\log_{10}(1) = 0 \quad \text{dB}. \tag{12}$$

So, at low frequencies, the magnitude response will approach 0 dB, i.e. a straight line of zero slope. As for high frequencies,

$$\begin{aligned} |T(x)|_{\text{dB, High Frequency}} &= 10\log_{10}\frac{1}{(RC\times 10^x)^2} = -20\log_{10}|RC\times 10^x| \\ &= -20\log_{10}|RC| - 20\log_{10}10^x \\ &= -20\log_{10}|RC| - 20x \quad \text{dB}. \end{aligned} \tag{13a}$$

The first term in the latter equation is a constant. Using $a = -20\log_{10}|RC|$ to rewrite this equation results in

$$|T(x)|_{\text{dB, High Frequency}} = a - 20x \quad \text{dB}, \tag{13b}$$

which has the familiar form $y = a + bx$ of a line with slope $b = -20$! **This is the famous -20 dB per decade slope you have been hearing all along.** When the frequency increases by a factor 10 (or a decade), x increases by 1 value and the magnitude response decreases by 20 dB.

Let me summarize what we have done so far: We found the transfer function in the Laplace and Fourier domains, found the magnitude and phase responses which are together called the frequency response, found the expression used in plotting Bode diagrams, and finally found its low- and high-frequency asymptotes. We noticed that for this first-order linear circuit these asymptotic approximations are two lines, one with zero slope which shows the response of the circuit at low-frequencies and the other with -20 dB/decade slope which approximates its behavior at high-frequencies. To make sure that we did not get lost amidst the mathematical elaborations, in Figure 3 I have laid on top of the magnitude response of our circuit, the low-frequency approximation of Eqn.(12) and the high-frequency approximation of Eqn.(13). You can see that they do indeed approximate the magnitude response at low and high frequencies.

As a final remark, let me talk a little bit about the question raised earlier in the discussion leading to Eqn.'s (10) and (11): For the system of Figure 1, which frequencies are considered *high* and which frequencies are considered *low*? The answer lies in the simplification process we followed. In the denominator of the amplitude response of Eqn.(7a), we have $1 + (\omega RC)^2$ and the asymptotic approximation we made was based on the comparison of ωRC with 1: $\omega RC \ll 1$ for low frequencies and $\omega RC \gg 1$ for high frequencies. Dividing both sides of these conditional expressions by RC, they could be expressed equivalently as

$$\text{Low Frequencies} : \omega \ll \frac{1}{RC} \tag{14a}$$

$$\text{High Frequencies} : \omega \gg \frac{1}{RC}. \tag{14b}$$

Let's denote the fraction on the right-hand side of these conditional expressions by ω_c: $\omega_c = \frac{1}{RC}$. (You will see why I chose the subscript c in a minute.) This is the frequency which is defined by the elements in our circuit, namely the resistance of the resistor R and the capacitance of the capacitor C, and is our reference for judging if a given frequency is low or high enough so we can use the approximations of Eqn.'s (10) or (11). (Note that it has the correct unit of frequency.) You can rewrite all the previous expressions in terms of ω_c. For instance, the transfer function in Eqn.(1) could be rewritten using ω_c as $T(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1+s/\omega_c}$. Using Eqn.(7a), you see that at this frequency

$$|T(\omega = \omega_c)|^2 = \frac{1}{1 + (\omega/\omega_c)^2}|_{\omega=\omega_c} = \frac{1}{1+1} = \frac{1}{2}. \tag{15}$$

We saw that the maximum of the magnitude response is 1 (see Eqn.(10) or Figure 2). So, ω_c is the frequency at which the *square* of the magnitude response is *half its maximum value.* Since beyond this point the square of the magnitude response, which is a measure of the *power* at each frequency, drops to less than a half of its maximum, this frequency is called the **cut-off frequency**, hence the subscript c. Since $10 \times \log_{10} \frac{1}{2} = -3\text{dB}$, this frequency is also called the **3 dB cut-off frequency**.

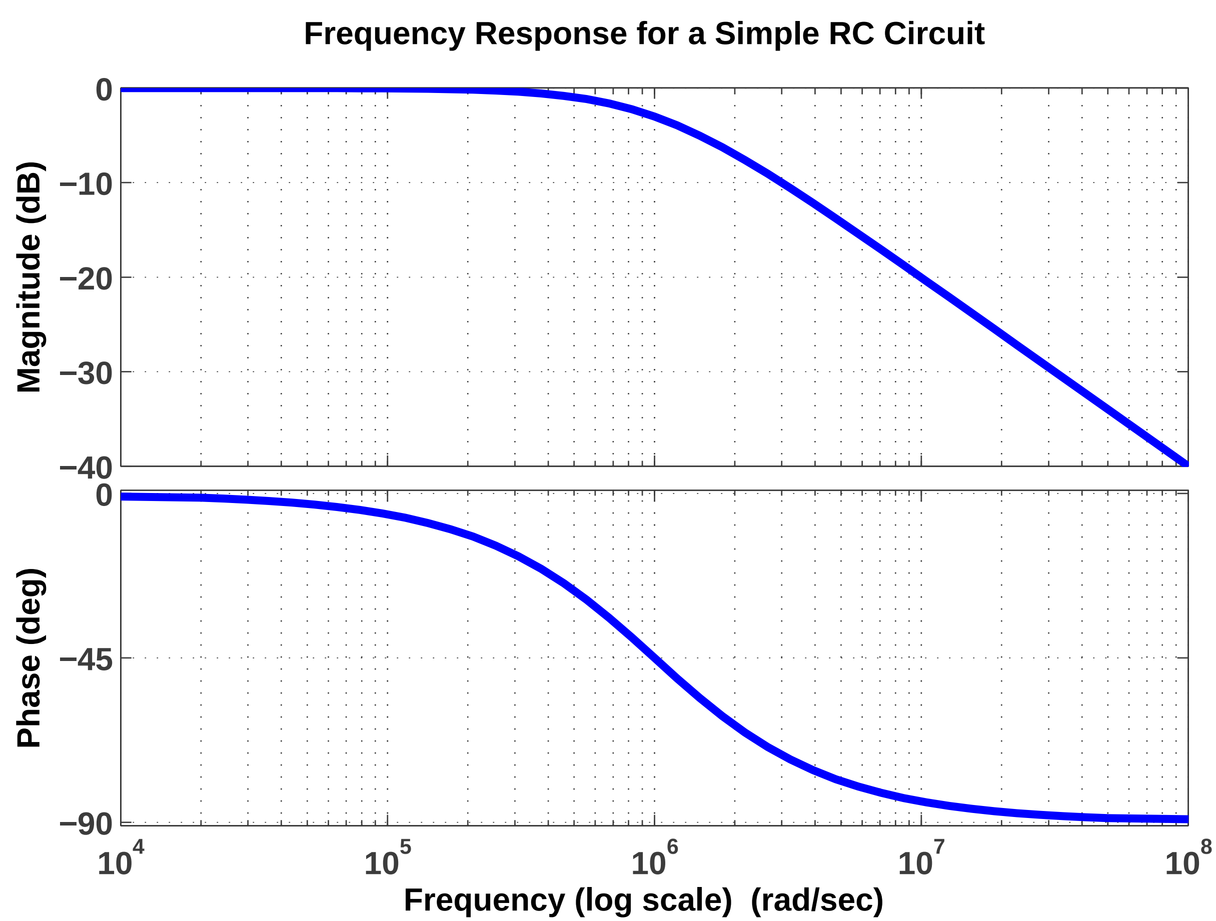

Figure 2: The Bode plot of the frequency response of the system in Figure 1.

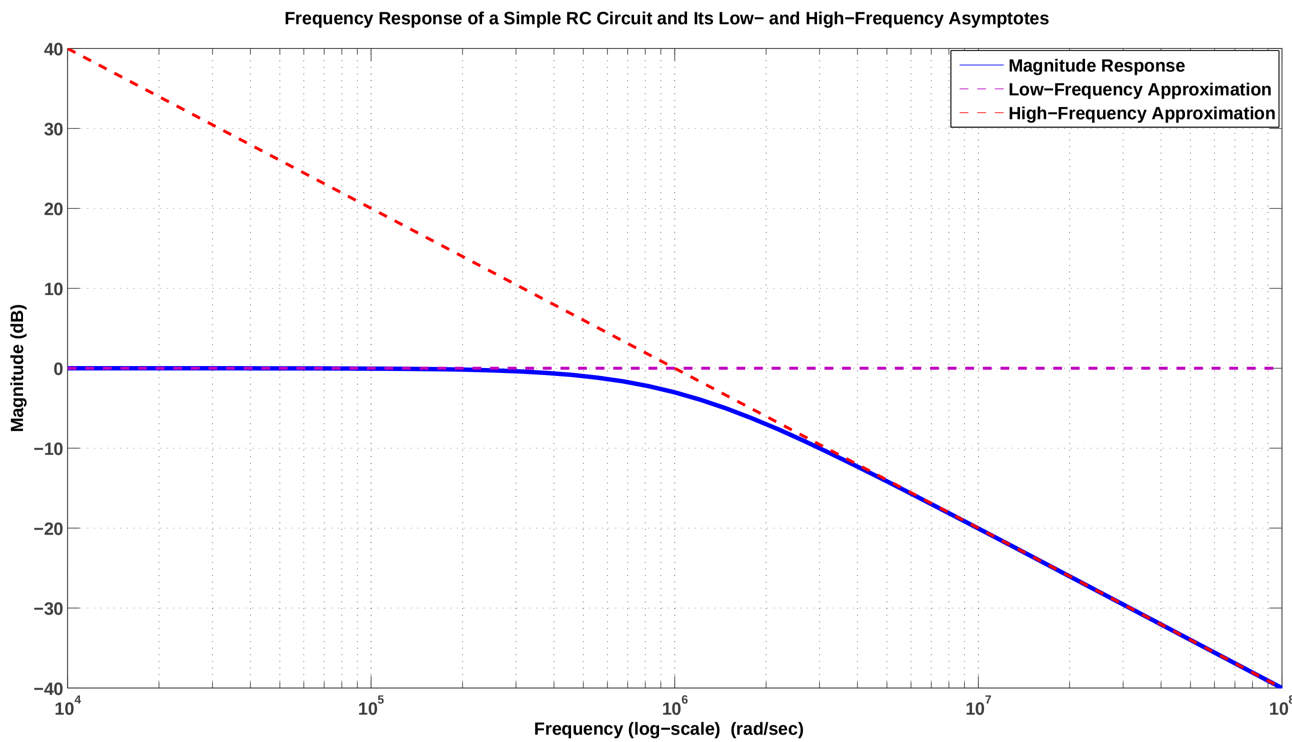

Figure 3: The Bode plot of the frequency response of the system in Figure 1 and its low- and high-frequency approximations. These asymptotes are plots of Eqn.(12), in magenta, and Eqn.(13), in red. Notice that these two lines cross extactly at $\omega = \omega_c$. This can be verified by simultaneous solution of Eqn.'s (12) and (13) for the frequency where they cross.